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MULTIVARIATE DEPENDENT RELEVATIONS WITH APPLICATIONS, AND RELATED TOPICS

by

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Abstract

"Relevation" is the name given by Krakowski (1973) (*Rev. Française Automat. Inform. Recherche Op.*, *7*, Ser. V-2, 108-120) to the distribution of failure time of a replacement from an aging stock. The present authors (Johnson and Kotz (~~1979; 1981~~) (*IEEE Trans. Reliability*, *R-28*, 292-299; *American Journal of Mathematical and Management Sciences*, *1*(2)) have extended this concept to include (i) hierarchal replacement systems and (ii) dependence between lifetimes of original and replacement items. In this paper, we present some further developments, including first steps towards a synthesis of (i) and (ii).

Key Words and Phrases: Relevation; Dependence; Aging in Stock; Aging in Service; Farlie-Gumbel-Morgenstern Distribution; Hierarchal Systems.

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1. Introduction.

The concept of a "relevation" was introduced by Krakowski (1973) to correspond to replacement from aging stocks. Suppose that when an element with survival distribution function (SDF) $S_1(t)$ fails, it is replaced by an element with SDF $S_2(t)$ taken from stock. If failure occurs at time t_1, then the conditional SDF of the replacement element is

$$S_{2|1}(t|t_1) = S_2(t)/S_2(t_1) \ ,$$

and the SDF of time of failure (T) of the replacement element is

$$S_T(\tau) = S_1(\tau) + S_2(\tau) \int_0^\tau f_1(t)\{S_2(t)\}^{-1}\, dt \ , \tag{1}$$

where $f_1(t) = -dS_1(t)/dt$ is the probability density function (PDF) corresponding to $S_1(t)$.

$S_T(\tau)$ is called the *relevation* of $S_1(t)$ with $S_2(t)$.

This concept has been extended to situations wherein there is a hierarchal system of replacement, and wherein the lifetimes of elements are not independent (Johnson and Kotz (1979, 1981)). In the present paper, we study combinations of extensions of this kind, and also give examples in which differential rates of aging in storage and in service are allowed for.

2. Dependent Relevations in a Hierarchal System.

Consider a set of n elements $\{\xi_1\}$, with a simple replacement element ξ_2. This might occur, for example, when ξ_2 is called into service as a back-up only when there is a failure among $\{\xi_1\}$. This system (comprised of $\{\xi_1\}$ and ξ_2) would quite possibly be only a small part of a complete hierarchal system; study of such relatively simple systems is an essential prerequisite to study of more elaborate systems of which they could be a part.

We denote the failure times of the elements of $\{\xi_1\}$ by $T_{11}, T_{12}, \ldots, T_{1n}$, and of ξ_2 by T_2. The joint survival distribution (SDF) of these $(n+1)$ failure times is

$$S_{\underset{\sim}{1}2}(\underset{\sim}{t}_1;t_2) = P\Big(\bigcap_{j=1}^{n}(T_{1j}>t_{1j}) \cap (T_2>t_2)\Big) \tag{2}$$

$(\underset{\sim}{t}_1 = (t_{11}, \ldots, t_{1n}))$. The joint SDF of $T_1 = \min(T_{11}, \ldots, T_{1n})$ -- the failure time of the first member of $\{\xi_1\}$ to fail -- is

$$S_{12}(t_1;t_2) = S_{\underset{\sim}{1}2}(t_1 \underset{\sim}{1}';\ t_2) \tag{3}$$

where $\underset{\sim}{1}' = (1,1,\ldots,1)$; and the SDF of T_1 is

$$S_1(t_1) = S_{\underset{\sim}{1}2}(t_1 \underset{\sim}{1}';\ 0)\ . \tag{4}$$

The SDF of the replacement element is then

$$S_T(\tau) = S_1(\tau) + \int_0^\tau f_1(t)\,\frac{S_{2|1}(\tau|t)}{S_{2|1}(t|t)}\,dt\ , \tag{5}$$

where $f_1(t) = -dS_1(t)/dt$ and $S_{2|1}(\tau|t) = P(T_2>\tau|T_1=t)$.

In general, explicit evaluation of (5) leads to rather complex formulas. As a compromise between realism and simplicity, we will (similarly as in Johnson and Kotz (1981)) use a generalized Farlie-Gumbel-Morgenstern distribution (Johnson and Kotz (1975)) as a special case for detailed examination. We suppose

$$S_{\underline{T}_1 2}(\underline{t}_1;t_2) = \{S_2(t_2)\prod_{j=1}^{n} S(t_{1j})\}[1 + \alpha\{1 - S_2(t_2)\}\sum_{j=1}^{n}\{1 - S(t_{1j})\}$$

$$+ \beta\sum_{j<j'}^{n-1}\sum^{n}\{1 - S(t_{1j})\}\{1 - S(t_{1j'})\}] \; . \qquad (6)$$

The parameters α and β have to satisfy certain conditions for the distribution to be a proper SDF. These conditions can be summarized as

$$|(2r-n)\alpha| - \tfrac{1}{2}\{(2r-n)^2 - n\}\beta \le 1 \qquad (r = 0,1,\ldots,n) \; ,$$

or equivalently, (7)

$$r'|\alpha| - \tfrac{1}{2}(r'^2-n)\beta \le 1 \qquad \begin{pmatrix} r' = 0,2,\ldots,n \text{ for } n \text{ even} \\ r' = 1,3,\ldots,n \text{ for } n \text{ odd} \end{pmatrix} .$$

The marginal SDF of each T_{1j} is $S(t)$; the marginal SDF of T_2 is $S_2(t)$. The joint SDF of $T_{11},\ldots,T_{1n}$ is

$$S_{\underline{T}_1}(\underline{t}_1) = \{\prod_{j=1}^{n} S(t_{1j})\}[1 + \beta\sum_{j<j'}^{n-1}\sum^{n}\{1 - S(t_{1j})\}\{1 - S(t_{1j'})\}] \; , \qquad (8)$$

which is an exchangeable distribution. This is likely to be a reasonable assumption if there are no systematic differences among the conditions to which the n elements of $\{\xi_1\}$ are exposed.

The SDF of T_1 (time to first failure in $\{\xi_1\}$) is

$$S_1(t_1) = \{S(t_1)\}^n \; [1 + \tfrac{1}{2}n(n-1)\beta\{1 - S(t_1)\}^2] \; , \qquad (9)$$

and the corresponding density function (PDF) is

$$f_1(t_1) = nf(t_1)\{S(t_1)\}^{n-1}\ [1 + \tfrac{1}{2}(n-1)\beta\{n - 2(n+1)S(t_1) + (n+2)[S(t_1)]^2\}]$$
$$= nf(t_1)\{S(t_1)\}^{n-1}\ [1 + \tfrac{1}{2}(n-1)\beta(1 - S(t_1))\{n - (n+2)S(t_1)\}]\ . \qquad (10)$$

The joint SDF of T_1 and T_2 is

$$S_{12}(t_1,t_2) = S_2(t_2)\{S(t_1)\}^n\ [1 + n\alpha\{1 - S(t_1)\}\{1 - S_2(t_2)\}$$
$$+ \tfrac{1}{2}n(n-1)\beta\{1 - S(t_1)\}^2]\ , \qquad (11.1)$$

with joint PDF

$$f_{12}(t_1,t_2) = nf_1(t_1)f_2(t_2)\{S(t_1)\}^{n-1}\ [1 + \alpha\{n - (n+1)S(t_1)\}\{1 - 2S_2(t_2)\}$$
$$+ \tfrac{1}{2}(n-1)\beta\{1 - S(t_1)\}\{n - (n+2)S(t_1)\}]\ . \qquad (11.2)$$

The conditional PDF of T_2, given T_1, is

$$f_{2|1}(t_2|t_1) = \frac{f_{12}(t_1,t_2)}{f_1(t_1)}$$
$$= f_2(t_2)\,\frac{1 + \alpha\{n - (n+1)S(t_1)\}\{1 - 2S_2(t_2)\} + \tfrac{1}{2}(n-1)\beta\{1 - S(t_1)\}\{n - (n+2)S(t_1)\}}{1 + \tfrac{1}{2}(n-1)\beta\{1 - S(t_1)\}\{n - (n+2)S(t_1)\}} \qquad (12.1)$$

and the conditional SDF is

$$S_{2|1}(t_2|t_1) = S_2(t_2)\,\frac{1 + \alpha\{n - (n+1)S(t_1)\}\{1 - S_2(t_2)\} + \tfrac{1}{2}(n-1)\beta\{1 - S(t_1)\}\{n - (n+2)S(t_1)\}}{1 + \tfrac{1}{2}(n-1)\beta\{1 - S(t_1)\}\{n - (n+2)S(t_1)\}} \qquad (12.2)$$

Hence from (5) the SDF of time to failure (T) of the ξ_2 (replacement) element is

$$S_T(\tau) = \{S(\tau)\}^n\,[1 + \tfrac{1}{2}n(n-1)\beta\{1 - S(\tau)\}^2]$$
$$+ n\int_0^\tau f(t)\{S(t)\}^{n-1}\,[1 + \tfrac{1}{2}(n-1)\beta\{1 - S(t)\}\{n - (n+2)S(t)\}]$$
$$\times \frac{S_2(\tau)}{S_2(t)}$$
$$\times \frac{1 + \alpha\{1 - S_2(\tau)\}\{n - (n+1)S(t_1)\} + \tfrac{1}{2}(n-1)\beta\{1 - S(t_1)\}\{n - (n+2)S(t_1)\}}{1 + \alpha\{1 - S_2(t)\}\{n - (n+1)S(t_1)\} + \tfrac{1}{2}(n-1)\beta\{1 - S(t)\}\{n - (n+2)S(t)\}}\,dt\;. \tag{13.1}$$

As a further specialization, we will take $S_2(t) = S(t)$. Then making the substitution $s = S(t)$ in the integral leads to

$$S_T(\tau) = \{S(\tau)\}^n\,[1 + \tfrac{1}{2}n(n-1)\beta\{1 - S(\tau)\}^2]$$
$$+ nS(\tau)\int_{S(\tau)}^1 s^{n-2}[1 + \tfrac{1}{2}(n-1)\beta(1-s)\{n - (n+2)s\}]$$
$$\times \frac{1 + \alpha\{1 - S(\tau)\}\{n - (n+1)s\} + \tfrac{1}{2}(n-1)\beta(1-s)\{n - (n+2)s\}}{1 + \alpha(1-s)\{n - (n+1)s\} + \tfrac{1}{2}(n-1)\beta(1-s)\{n - (n+2)s\}}\,ds\;. \tag{13.2}$$

Note that $S_T(\tau)$ is always the same function of $S(\tau)$, whatever the functional form of the latter.

The integral in (13.2) can be evaluated in terms of elementary functions (see Appendix), though the expression is, in general, rather complicated. For the special case $\alpha = 0$ (so that replacement lifetime is independent of the lifetimes of the initial elements),

$$S_T(\tau) = \{S(\tau)\}^n\ [1 + \tfrac{1}{2}n(n-1)\beta\{1 - S(\tau)\}^2]$$

$$+\ nS(\tau) \int_{S(\tau)}^{1} s^{n-2}[1 + \tfrac{1}{2}(n-1)\beta(1-s)\{n - (n+2)s\}]\,ds$$

$$= \frac{S(\tau)[n - \{S(\tau)\}^{n-1}]}{n-1}$$

$$+\ \frac{\beta S(\tau)}{n+1}[1 - \tfrac{1}{2}\{S(\tau)\}^{n-1}\ [n(n+1) - 2(n^2-1)S(\tau) + n(n-1)\{S(\tau)\}^2]]$$

$$= A_n + B_n\beta\ . \qquad (14)$$

Some numerical values of A_n and B_n are shown in Table 1.

TABLE 1

Values of A_n and B_n for $S_T(\tau) = A_n + B_n\beta$ when $\alpha = 0$.

$S(\tau)$ \ n =	2		3		4		5	
	A_n	B_n	A_n	B_n	A_n	B_n	A_n	B_n
0.2	0.360	0.0341	0.296	0.0410	0.266	0.0377	0.250	0.0328
0.4	0.640	0.0288	0.568	0.0475	0.524	0.0546	0.497	0.0547
0.6	0.840	0.0128	0.792	0.0269	0.757	0.0381	0.731	0.0487
0.8	0.960	0.0021	0.944	0.0054	0.930	0.0094	0.918	0.0132

(NOTE: Possible values of β are limited by $-2n^{-1}(n-1)^{-1} \le \beta \le 2n^{-1}$.)

If $\alpha \neq 0$, the value of $S_T(\tau)$ exceeds that given in (14) by

$$n\alpha S(\tau) \int_{S(\tau)}^{1} \frac{s^{n-2}[1+\tfrac{1}{2}(n-1)\beta(1-s)\{n-(n+2)s\}]\{s - S(\tau)\}\{n-(n+1)s\}}{1+\alpha(1-s)\{n-(n+1)s\}+\tfrac{1}{2}(n-1)\beta(1-s)\{n-(n+2)s\}}\,ds\ .$$

If $S(\tau) > n/(n+1)$, this will be of opposite sign to α since

$$\{s - S(\tau)\}\{n - (n+1)s\} < 0 \quad \text{for } S(\tau) < s < 1\ .$$

This gives some qualitative appreciation of the effect of non-zero α.

To see how these results can be used in dealing with more complicated systems, consider the subsystem set out schematically in Figure 1. The

ξ_3

ξ_2

ξ_1

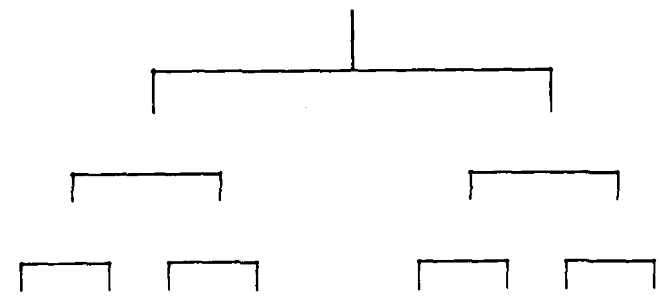

FIGURE 1

component ξ_3 is related to the two ξ_2's in the same way as each ξ_2 is related to the two ξ_1's belonging to it. ξ_3 *comes into operation* at the first failure of the two ξ_2's. The relevation SDF $S_T(\tau)$ given by (13.1) now plays the role previously played by the SDF of each ξ_1 $(S(t))$. Of course, if the resulting formula for the SDF of the ξ_3 replacement were to be written out explicitly, it would be very lengthy indeed. However, the computation of numerical values is quite straightforward. It can be carried out in stages: first finding $S_T(\tau)$ and then combining it with the SDF -- $S_3(t)$, say -- of the ξ_3 components. (The case of independent components has been treated by Johnson and Kotz (1979).)

3. Differential Rates of Aging in Storage and in Service.

It is likely that an element will be under less stress while in storage than in service. This can be allowed for by regarding a period t in storage as equivalent to a period $g(t)$ in service (usually with $g(t) < t$). Equation (1)

is then replaced by

$$S_T(\tau) = S_1(\tau) + \int_0^\tau f_1(t) \frac{S_2(g(t) + \tau - t)}{S_2(g(t))} dt , \tag{15}$$

and (2) by

$$S_T(\tau) = S_1(\tau) + \int_0^\tau f_1(t) \frac{S_{2|1}(g(t) + \tau - t \,|\, t)}{S_{2|1}(g(t) \,|\, t)} dt . \tag{16}$$

One might reasonably take $g(t) = \gamma t$ -- so that 1 year in storage is equivalent to γ years in service. Usually one would take $\gamma < 1$, but this is not essential. Even if it is assumed that $S_1(t) = S_2(t) = S(t)$, it is no longer true that $S_T(\tau)$ depends only on $S(\tau)$. As an example, we take $\gamma < 1$ and

$$S(t) = 1 - \kappa t \qquad (0 \le t \le \tau \le \kappa^{-1}) .$$

Then from (15),

$$\begin{aligned} S_T(\tau) &= 1 - \kappa\tau + \int_0^\tau \kappa \frac{1 - \kappa\{\tau - (1-\gamma)t\}}{1 - \kappa\gamma t} dt \\ &= 1 - \kappa\tau\gamma^{-1} - (1 - \kappa\tau\gamma)\gamma^{-2} \log(1 - \kappa\tau\gamma) . \end{aligned} \tag{17}$$

Some numerical values are given in Table 2.

TABLE 2

Values of $S_T(\tau)$ when $S(\tau) = 1 - \kappa\tau$ has specified values (γ = service years equivalent to one year in storage).

$S(\tau) \backslash \gamma$	0.2	0.4	0.6	0.8	1.0
0.2	0.661	0.639	0.611	0.562(5)	0.522
0.4	0.812	0.804	0.794	0.781	0.767
0.6	0.918	0.915	0.913	0.910	0.906
0.8	0.980	0.979	0.979	0.979	0.979

(The last column corresponds to no difference between aging in storage and in service.) The effect of differential aging is not very great, especially at the longer durations.

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References

Johnson, N.L. and Kotz, S. (1975). On some generalized Farlie-Gumbel-Morgenstern distributions. *Comm. Statist.*, *4*, 415-427.

Johnson, N.L. and Kotz, S. (1979). Models of hierarchal replacement. *IEEE Trans. Reliability*, *R-28*, 292-299.

Johnson, N.L. and Kotz, S. (1981). Dependent relevations: Time-to-failure under dependence. *American Journal of Mathematical and Management Sciences*, *1*(2).

Krakowski, M. (1973). The relevation transform and a generalization of the gamma distribution function. *Rev. Française Automat. Inform. Recherche Op.*, *7*, Ser. V-2, 108-120.

Appendix

The integrand in (13.2) can be expressed in the form

$$[a_n\{(s-b_n)^2+c_n\}]^{-1}\sum_{j=0}^{4} d_{n,j}\, s^{n+j-2}\ .$$

Using the change of variable $y = s - b_n$, the integral becomes

$$a_n^{-1}\int_{S(\tau)-b_n}^{1-b_n}\{\sum_{j=0}^{4} d_{n,j}(y+b_n)^{n-2+j}\}(y^2+c_n)^{-1}\,dy$$

$$= a_n^{-1}\sum_{j=0}^{4} d_{n,j}\sum_{h=0}^{n-2+j}\binom{n-2+j}{h} b_n^{n-2+j-h}\int_{S(\tau)-b_n}^{1-b_n} y^h(y^2+c_n)^{-1}\,dy$$

$$= a_n^{-1}b_n^{n-2}\sum_{j=0}^{4} d_{n,j}b_n^{j}\sum_{h=0}^{n-2+j}\binom{n-2+j}{h} b_n^{-h}[I_h(1-b_n) - I_h(S(\tau)-b_n)]\ ,$$

where

$$I_h(y) = \int_0^y y^h(y^2+c_n)^{-1}\,dy$$

$$= \sum_{i=0}^{[(h-3)/2]}\frac{(-c_n)^i}{h-1-2i} + \begin{cases} \tfrac12(-c_n)^{\frac12(h-1)}\log(y^2+c_n) & \text{(h odd)} \\ (-1)^{\frac12 h}\, c_n^{\frac12(h-1)}\tan^{-1}(y/\sqrt{c_n}) & \text{(h even, } c_n > 0) \\ \tfrac12(-c_n)^{\frac12(h-1)}\log\dfrac{y-\sqrt{-c_n}}{y+\sqrt{-c_n}} & \text{(h even, } c_n < 0)\ . \end{cases}$$

[For example, $I_0(y) = 1/\sqrt{c_n}\,\tan^{-1} y/\sqrt{c_n}$ if $c_n > 0$; $[1/(2\sqrt{-c_n})]\log((y-\sqrt{-c_n})/(y+\sqrt{-c_n}))$ if $c_n < 0$; $I_1(y) = \tfrac12\log(y^2+c_n)$; $I_2(y) = y - c_nI_0(y)$; $I_3(y) = \tfrac12 y^2 - c_nI_1(y)$; $I_4(y) = (1/3)y^3 - c_nI_2(y)$; and so on.]

The constants a_n, b_n, and c_n are:

$$a_n = (n+1)\alpha + \tfrac{1}{2}(n-1)(n+2)\beta \;;$$

$$b_n = \tfrac{1}{2}\{(2n+1)\alpha + (n^2-1)\beta\}a_n^{-1} \;;$$

$$c_n = [a_n - 1/4\{\alpha + (n-1)\beta\}^2]a_n^{-2} \;.$$

(Note that if $\alpha = 0$, $a_n = \tfrac{1}{2}(n-1)(n+2)\beta$; $b_n = (n+1)/(n+2)$; $c_n = 2\{(n-1)(n+2)\beta\}^{-1} - (n+2)^{-2}$.)

The expressions for the coefficients $d_{n,j}$ are:

$$d_{n,0} = \{1 + \tfrac{1}{2}n(n-1)\beta\}[1 + n\alpha\{1 - S(\tau)\} + \tfrac{1}{2}n(n-1)\beta] \;;$$

$$d_{n,1} = -2(n^2-1)\{1 + \tfrac{1}{2}n(n-1)\beta\}\beta - (n+1)\alpha\{1 + (3/2)n(n-1)\beta\}\{1 - S(\tau)\} \;;$$

$$d_{n,2} = (n-1)(n+2)\beta + \tfrac{1}{2}(n-1)(3n^2+6n+2)[\alpha\{1 - S(\tau)\} + (n-1)\beta]\beta \;;$$

$$d_{n,3} = -(n^2-1)(n+2)[\tfrac{1}{2}\alpha\{1 - S(\tau)\} + (n-1)\beta]\beta \;;$$

$$d_{n,4} = (1/4)(n-1)^2\,(n+2)^2\,\beta^2 \;.$$

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"Relevation" is the name given by Krakowski (1973) (*Rev. Française Automat. Inform. Recherche Op.*, *7*, Ser. V-2, 108-120) to the distribution of failure time of a replacement from an aging stock. The present authors (Johnson and Kotz (1979; 1981) (*IEEE Trans. Reliability*, *R-28*, 292-299; *American Journal of Mathematical and Management Sciences*, *1*(2))) have extended this concept to include (i) hierarchal replacement systems and (ii) dependence between lifetimes of original and replacement items. In this paper, we present some further developments, including first steps towards a synthesis of (i) and

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